progression; or that the sum of their squares shall be a square, or that they shall be the sides of a right-angled triangle. In fact the number of requirements is unlimited.

In the first case, let $a = 2rs - r^2 + s^2$, $b = r^2 + s^2$, $c = 2rs + r^2 - s^2$, which are general expressions for three square numbers in arithmetical progression; r, s, being any different numbers that will make a, b, c, all positive. Let $r = 2$, $s = 1$; then $a = 1$, $b = 5$, $c = 7$, and

$$x = \frac{151321}{7863240}. \quad \therefore \frac{151321}{7863240}, \frac{756605}{7863240}, \text{ and } \frac{1059247}{7863240}$$

are the roots for the positive sign in $a^2x^2 \pm ax = \square$, &c. In order to have a positive value of x to satisfy the negative sign, take $r = 4$, $s = 3$; then $a=17$, $b=25$, $c=31$, and $x = \dfrac{(864571)^2}{11011044931800}$. Therefore

$$\frac{12707211238697}{11011044931800}, \frac{18687075351025}{11011044931800}, \frac{23171973435271}{11011044931800},$$

are the roots for the negative sign in $a^2x^2 \pm ax = \square$, &c.

SOLUTIONS OF PROBLEMS IN NUMBER 2.

101.—"From any point O, within the circumference of a circle, two lines are drawn making a constant angle with each other. These lines revolve about O in the plane of the circle, and from the points where they cut the circumference tangents are drawn. Find the locus of the intersection of these tangents.

SOLUTION BY PROF. W. W. BEMAN, ANN ARBOR, MICHIGAN.

LET a diameter through O represent the axis of X. Put $r =$ the radius of the given circle, $a =$ the distance of the point O from its centre, $\alpha =$ the constant angle included by the two given lines; x, y, coordinates of the intersection of the tangents, the origin being at the centre of the circle, and x', y', and x'', y'', coordinates of the points where the given lines cut the circumference of the given circle. Then we easily obtain:

$$xx' + yy' = r^2 \ldots . (1); \qquad xx'' + yy'' = r^2; \ldots\ldots (2)$$

$$x'^2 + y'^2 = r^2 \ldots . (3); \qquad x''^2 + y''^2 = r^2; \ldots\ldots (4)$$

$$(x' - x'')^2 + (y' - y'')^2 = (x' - a)^2 + y'^2 + (x'' - a)^2 + y''^2$$
$$- 2\sqrt{\{[(x' - a)^2 + y'^2][(x'' - a)^2 + y''^2]\}} \cos\alpha; \text{ or,}$$

$$x'x'' + y'y'' - a(x' + x'') + a^2 = \sqrt{[(r^2 + a^2 - 2ax')(r^2 + a^2 - 2ax'')]} \cos\alpha \ldots (5)$$

Combining (1) and (3),

$$x' = \frac{r}{x^2 + y^2}\left(rx \pm y\sqrt{(x^2 + y^2 - r^2)}\right); \; y' = \frac{r}{x^2 + y^2}\left(ry \mp x\sqrt{(x^2 + y^2 - r^2)}\right).$$

Combining (2) and (4),

$$x'' = \frac{r}{x^2+y^2}\Big(rx\mp y\sqrt{(x^2+y^2-r^2)}\Big);\ y'' = \frac{r}{x^2+y^2}\Big(ry\pm x\sqrt{(x^2+y^2-r^2)}\Big).$$

Substituting these values in (5), and reducing,

$$[2r^4-(r^2-a^2)(x^2+y^2)-2ar^2x]^2=[(r^2+a^2)^2(x^2+y^2)^2-4ar^2(r^2+a^2)(x^2+y^2)x \\ +4a^2r^2(r^2-y^2)(x^2+y^2)]\cos^2\alpha;$$

or, in a better form for discussion,

$$[2r^4-(r^2-a^2)(x^2+y^2)-2ar^2x]^2\tan^2\alpha = 4r^2(x^2+y^2-r^2)(ax-r^2)^2.$$

When $\alpha=0$, we have $x^2+y^2-r^2=0$, and $(ax-r^2)^2=0$, the original circle, and two coincident straight lines. When $\alpha=90°$, $[2r^4-(r^2-a^2)\times(x^2+y^2)-2arx^2]^2=0$, two coincident circles.

As α increases from 0, the circle passes into an oval with the horizontal axis the longer, while the two straight lines separate at first into two infinite branches. The right hand branch recedes to infinity and then reappears on the left, forming now with the other line an oval whose vertical axis is the longer. These ovals then approach till, when $\alpha = 90°$, they coincide in a circle.

When $a=0$, the equation becomes

$$(x^2+y^2-r^2\operatorname{cosec}^2\tfrac{1}{2}\alpha)(x^2+y^2-r^2\sec^2\tfrac{1}{2}\alpha)=0,$$

two circles, coincident when $\alpha = 90°$, as should be expected.

When $a=r$, the equation becomes $(x^2+y^2-r^2\sec^2\alpha)(x-r)^2=0$, a circle and two coincident straight lines.

The locus can be easily constructed for a number of values of α at the same time, and the interesting changes seen at a glance.

[Dr. Eggers gets

$$2(px+qy-r^2)\sqrt{(x^2+y^2-r^2)}=[2(px+qy-r^2)+(r^2-p^2-q^2)(x^2+y^2)]\tan\alpha,$$

for the required equation; in which p and q are coordinates of the point O.

Mr. Seitz gets

$$\cos\theta = \frac{r^2}{a\rho} - \frac{(r^2-a^2)\rho\tan\alpha}{2ar[r\tan\alpha-\sqrt{(\rho^2-r^2)}]},$$

for the polar equation of the curve; where ρ is the radius vector and θ the angle between the radius vector and a diameter through O.]

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102. "Prove that in every triangle the square of the sum of the squares of the sides exceeds double the sum of their fourth powers."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

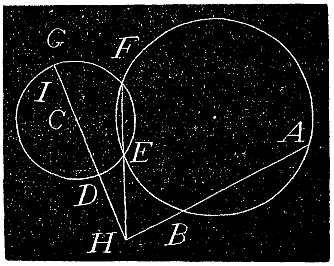
LET a, b, c be the sides. Then $a+b-c>0$, $a+c-b>0$, $b+c-a>0$, and $a+b+c>0$. By multiplication, $-a^4-b^4-c^4+2a^2b^2+2a^2c^2+2b^2c^2>0$. Adding to both memb's $2(a^4+b^4+c^4)$, we have $(a^2+b^2+c^2)^2>2(a^4+b^4+c^4)$.

103. "Through two given points draw a circle bisecting the circumference of a given circle."

SOLUTION BY HENRY HEATON, DESMOINES, IOWA.

LET A and B represent the given points and GFE the given circle.

Through A and B describe any circle $ABEF$, intersecting the given circle in E and F. Draw EF and prolong it until it meets AB produced in H. Through the center C, draw HG cutting the given circle in D and G; then will D and G be points in the required circle. For $AH \times BH = FH \times EH = GH \times DH$; and if a circle be passed through the points A, B and D it will cut HG, or HG produced, at some point I; and we have $AH \times BH = IH \times DH = GH \times DH$. Hence I coincides with G, and the circle ABD passes through G.



[The solutions by Dr. Eggers, Prof. Johnson, Prof. M. C. Stevens and Mr. Seitz are substantially the same as the above. Prof. Stevens' solution was received too late to be included in our notice of solutions, published in No. 2. We have, also, an elegant solution of this question by Miss Ladd which should have been included with the notices in No. 2, but was omitted through an oversight.]

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104. "Show that $x^3 + 1 = y^2$ is possible for the values $x = 0, -1$, and 1, only."

SOLUTION BY PROF. A. B. EVANS, LOCKPORT, NEW YORK.

IF $x^2 + 1 = y^2$ we may put $x = m - 1$; then

$$x^3 + 1 = m^3 - 3m^2 + 3m = \square. \qquad (1)$$

It is evident that (1) must be divisible by m^2; put, therefore,

$$m^3 - 3m^2 + 3m = a^2m^2; \text{ then } m^2 - (a^2 + 3)m = -3, \qquad (2)$$

and

$$m = 0. \qquad (3)$$

From equation (2), $m = \frac{1}{2}(a^2 + 3) \pm \frac{1}{2}\sqrt{[(a^2 + 3)^2 - 12]}$. Hence

$$(a^2 + 3)^2 - 12 = \square.$$

Now $a^2 + 3 = 4$ is the only value of $(a^2 + 3)$ in integers that will satisfy this condition; $\therefore a = \pm 1$, $m = 2 \pm 1 = 3$ or 1, $x = m - 1 = 2$ or 0. From equation (3) $x = -1$.

[Dr. Nelson requests us to state that problems 104 and 105 were contributed as selected and not as original. The above solution shows that the value $x = 1$, &c., as announced, should have been $x = 0, -1$, and 2.]

105. "Show that in taking a handful of shot from a bag it is more probable that an odd number will be drawn than an even number."

SOLUTION BY PROF. H. T. J. LUDWICK, SALISBURY, N. C.

LET p = the probability that the number drawn will be even and
q = " " " " " " " " odd.

The number of shot in the bag is either even or odd, each of which cases is equally probable. If the number be even, then $q = p$, if odd, $q > p$. Hence, as q may be greater but never less than p, the probability is in favor of drawing an odd number.

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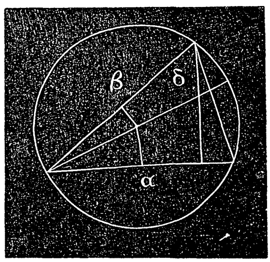
106 "Interpret $\dfrac{\alpha\beta^2 - \beta\alpha^2}{V\alpha\beta}$."

SOLUTION BY PROF. P. E. CHASE, PHILADELPHIA, PA.

$$\frac{\alpha\beta.\beta - \beta\alpha.\alpha}{V\alpha\beta} = \frac{S\alpha\beta(\beta-\alpha) + V\alpha\beta(\beta+\alpha)}{V\alpha\beta} = \frac{(\alpha-\beta)\cos\theta}{\varepsilon\sin\theta} + (\alpha+\beta)$$

= the diagonal of a parallelogram, of which one side is parallel to one diagonal of (α, β) and the other side is perpendicular to the other diagonal.

SOLUTION BY CHRISTINE LADD, UNION SPRINGS, N. Y.



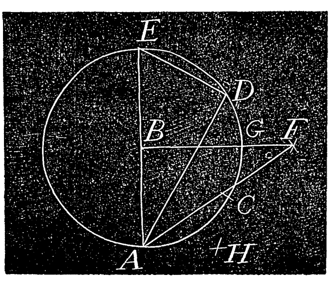
IF ρ be the radius of the circle which circumscribes the quaternion $\dfrac{\beta}{\alpha}$, we have, by similar triangles, $2\dfrac{\rho}{\beta} = \dfrac{\alpha-\beta}{\delta}$, in which δ is the perpendicular from the extremity of β upon α. But $\delta = -\alpha^{-1}V\alpha\beta$; hence,

$$2\rho = \frac{(\alpha-\beta)\beta}{-\alpha^{-1}V\alpha\beta} = \frac{\alpha^2\beta - \beta^2\alpha}{V\beta\alpha} = \frac{\alpha\beta^2 - \beta\alpha^2}{V\alpha\beta}.$$

If we put $\gamma = \alpha - \beta$, $2\rho = \dfrac{\alpha\gamma\beta}{V\beta\alpha} = \dfrac{\gamma\beta\alpha}{V\alpha\gamma} = \dfrac{\beta\alpha\gamma}{V\gamma\beta}$.

99.—[As several of our correspondents object to the published solution of this problem, as being impracticable, we add the following:]

SOLUTION BY PROF. M. C. STEVENS, SALEM, OHIO.



THE construction is sufficiently explained by the figure, in which $AC = CD = DE = AB$ = radius of the circle, A and B being the two given points and G and H the required points; $AF = EF = AD$ and $AG = BF$.

Demonstration. Because $AF^2 = AD^2 = 3AB^2 = AB^2 + BF^2$, $\therefore AG^2\ (=BF^2) = 3AB^2 - AB^2 = 2AB^2$, $\therefore \angle ABG$ is a right-angle.